WORKING PAPER
ALFRED P. SLOAN SCHOOL OF MANAGEMENT

Determinacy of Equilibrium in Infinite Horizon Economies: A Perspective

Walter J. Muller, III*

Number 1509-83 December 1983

MASSACHUSETTS
INSTITUTE OF TECHNOLOGY
50 MEMORIAL DRIVE
CAMBRIDGE, MASSACHUSETTS 02139

Determinacy of Equilibrium

in Infinite Horizon Economies:

A Perspective

Walter J. Muller, III*

Number 1509-83 December 1983

*The author wishes to thank Tim Kehoe, David Levine, Andreu Mas-Colell, Julio Rotemberg, and Michael Woodford for helpful comments.

I. Introduction

A desirable property for the equilibria of any economic model is local determinacy. That is, one would like the specification of endowments, technology, and preferences in the model to suffice to uniquely determine the equilibrium values of all economic variables, at least locally. This is necessary to perform the familiar sort of comparative statics exercises to determine the effect of parameter changes on equilibrium values. It is well known that the perfect foresight competitive equilibria of dynamic general equilibrium models need not possess this property. There are a number of examples in the literature showing overlapping generations economies with a continuum of equilibria. Recent work by Kehoe and Levine (1983b) shows that in stationary overlapping generations exchange economies the dimension of indeterminacy is limited only by the number of goods in each period. Muller and Woodford (1983b) have extended their results to stationary overlapping generations economies with production and infinite-lived agents.

These results have serious implications for the type of models that should be used in policy questions, but they may seem inaccessible because of the technical nature of the aforementioned papers. Accordingly, the present paper attempts to remedy this. The strategy is to first present an intuitive discussion of the work of Muller and Woodford (1983a, 1983b)

and then to provide a perspective for their results by relating them to the relevant papers in the overlapping generations literature, the optimal growth literature, and the macroeconomic theory literature. The paper concludes with a discussion of directions for future research.

II. A Survey of the Results of Muller and Woodford

Muller and Woodford develop a stationary overlapping generations model with production, with infinite-lived as well as finite-lived consumers, and with nondepreciating assets called "land" that yield a stationary stream of consumption goods. There are n goods in each time period. Each generation $t=1,2,3...$ of finite-lived agents is identical, and consists of a finite number of agents who consume in period t and t+1. There is also a generation 0 that consists of a finite number of agents who consume only in period 1. In addition, there is a finite set of infinite-lived agents that live in every period. Each infinite-lived agent has an additively separable utility function and an endowment that is the same in each period. Finally, there is a constant returns to scale production technology available in each period that allows production spanning two periods.

The consumption and savings decisions of the different types of consumers in generation t, for $t=1,2,3...$, are aggregated into excess demand functions $y(p_t, p_{t+1})$ in period t

and $z(p_t, p_{t+1})$ in period t+1 where p_t denotes the price vector for goods in period t. The aggregate excess demand of generation 0 is denoted $z_0(p_1)$. It is assumed that the excess demand functions have been generated by a set of utility maximizing consumers, that the excess demand functions are bounded below, and that as the price of any good goes to zero, the total excess demand for that good goes to infinity.

There are also a finite number of infinite-lived agents, denoted h=1,2,...,H. Infinite-lived consumer h has a utility function

(2.1) $$V^h = \sum_{t=1}^{\infty} (\beta_h)^{t-1} u^h(c_t^h)$$

where c_t^h is the consumption vector for h in period t and $0 \leq \beta_h < 1$ is the discount factor for h. The single period utility function u^h is assumed to be bounded, differentiable, monotonic, concave, and to satisfy two technical conditions that assure behavior is not perverse near zero consumption. Infinite-lived consumer h has an n-vector of endowments a^h in each period, where $a^h >> 0$. Given a price sequence $\{p_t\}_1^{\infty}$, the consumption demands of h are chosen so as to maximize V^h subject to the lifetime budget constraint

$$(2.2) \qquad \sum_{t=1}^{\infty} p_t' c_t^h < \sum_{t=1}^{\infty} p_t' a^h .$$

The production technology is specified in such a way that in any period all the profit opportunities of the infinite horizon economy can be characterized by a set of two-period profit functions that have the standard properties. Since the technology has constant returns to scale, the profit functions describe the maximum profit possible subject to the constraint that the netput vector of inputs and outputs has norm less than or equal to 1. Let $\Psi_j(p_t, p_{t+1})$ for j=1,2,...m, describe the production technology. We use the following notation for the derivatives of the profit functions.

$$B_k(p_t, p_{t+1})' = \begin{bmatrix} D_k\Psi_1(p_t, p_{t+1}) \\ \cdot \quad \cdot \quad \cdot \\ D_k\Psi_m(p_t, p_{t+1}) \end{bmatrix}$$

for k=1,2. Recall that for any differentiable profit function Hotelling's lemma says that the profit maximizing netput vector for any vector of prices is the gradient vector of the profit function. Given the technology has constant returns to scale, for any (p_t, p_{t+1}) the production possibilities of the economy can now be described by the set of all

$$\begin{bmatrix} B_1(p_t, p_{t+1}) \\ B_2(p_t, p_{t+1}) \end{bmatrix} b_t$$

for $b_t \geq 0$. That is, B_1 and B_2 play the role of an activity analysis matrix. In equilibrium it must be the case that all activities earn nonpositive profits, so one must have

$$p_t' B_1(p_t, p_{t+1}) + p_{t+1}' B_2(p_t, p_{t+1}) \leq 0$$

if (p_t, p_{t+1}) are to be equilibrium prices.

Any $(z_0, z, y, V^h, a^h, \Psi_j)$ of the sort described specifies a stationary overlapping generations economy. The asset I call land arises when an infinite-lived agent sells a claim entitling the holder to a portion of the infinite-lived agent's endowment in every period. An infinite-lived agent would sell such a claim to finance a higher level of current consumption at the expense of future consumption. The equilibrium concept of interest is now:

<u>Definition.</u> A perfect foresight equilibrium of economy $(z_0, z, y, V^h, a^h, \Psi_j)$ is a price sequence $\{p_t\}_1^\infty$, an activity sequence $\{b_t\}_1^\infty$, and consumption sequences $\{c_t^h\}_1^\infty$, for $h=1,2,\ldots,H$, such that for each $h, \{c_t^h\}_1^\infty$ maximizes (2.1) subject to (2.2),

$$(2.3) \quad z_0(p_1)+y(p_1,p_2) + \sum_{h=1}^{H} c_1^h - \sum_{h=1}^{H} a^h - B_1(p_1,p_2)b_1 = 0,$$

$$(2.4) \quad z(p_{t-1},p_t)+y(p_t,p_{t+1}) + \sum_{h=1}^{H} c_t^h - \sum_{h=1}^{H} a^h$$
$$-B_1(p_t,p_{t+1})b_t - B_2(p_{t-1},p_t)b_{t-1} = 0, \text{ for all}$$
$$t = 2,3,\ldots, \text{ and}$$

$$(2.5) \quad p_t'B_1(p_t,p_{t+1})+p_{t+1}'B_2(p_t,p_{t+1}) \leq 0, \ b_t \geq 0, \text{ and}$$
$$p_t'B_1(p_t,p_{t+1})b_t + p_{t+1}'B_2(p_t,p_{t+1})b_t = 0, \text{ for all}$$

$t=1,2,\ldots$

Equations (2.3) and (2.4) require that supply equal demand in each period, while (2.5) requires that all production activities yield non-positive profits in equilibrium and that the production plan actually in use make zero profits.

By extending the proof of Wilson (1981), Muller and Woodford easily establish the existence of perfect foresight equilibrium in a stationary overlapping generations model with production and infinite-lived consumers. They also examine the possibility of existence of monetary equilibria in their model. That is, the existence of equilibria in an economy in which there is a stock of fiat money in the hands of the initial old and there is no money creation or destruction thereafter. This is the familiar case in which the money stock represents the fixed nominal net savings of the economy for all time. Muller

and Woodford show that a monetary equilibrium is only possible if neither land nor infinite-lived consumers exist in the economy. Note that this last result depends crucially on the assumption of a stationary economy, for Wilson (1981) and Tirole (1983) have provided simple examples showing it is not true in nonstationary models.

Muller and Woodford largely restrict their attention to what happens near steady states of the economy. By a steady state is meant an equilibrium in which $p_t=\beta^t p, b_t=b, c_t^h=c^h$, for all t.

<u>Definition.</u> A perfect foresight steady state equilibrium of economy (z,y,V^h,a^h,Ψ_j) is a price vector p, a scalar β, an activity vector b, consumption vectors c^h for h=1,...,H, and non-negative scalars λ^h for each h such that $c^h \neq 0$, satisfying

$$(2.6)\quad z(p,\beta p)+y(p,\beta p)+\sum_{h=1}^{H} c^h -a -B_1(p,\beta p)b$$
$$-B_2(p,\beta p)b=0,$$

$$(2.7)\quad p'B_1(p,\beta p)+\beta p'B_2(p,\beta p)\leq 0,\ b\geq 0,$$

$$(2.8)\quad p'B_1(p,\beta p)b+\beta p'B_2(p,\beta p)b=0,$$

(2.9) $\beta_h \leq \beta < 1$ for $h=1,2,\ldots,H$, with $\beta_h = \beta$ for every h such that such that $c^h \neq 0$, and

(2.10) $Du^h(c^h) = \lambda^h p$ for every h such that $c^h \neq 0$.

Note that the possible steady states are determined solely by the stationary features of the economy. Therefore a steady state equilibrium is not an equilibrium in the sense described earlier unless the initial conditions are carefully chosen to be consistent with it, although other equilibrium paths may approach a steady state asymptotically.

Order the infinite-lived consumers so that $\beta_1 \geq \beta_2 \ldots \geq \beta_H$. Then (2.9) requires that $\beta \geq \beta_1$ at a steady state. The steady state β cannot be less than β_1 because if that were the case, the optimal consumption path for infinite-lived agent 1 would not be a constant consumption vector in each period. There are then two possible kinds of steady states: those in which $\beta > \beta_1$ and those in which $\beta = \beta_1$. In the former case the steady state is determined exactly as in an economy where there are no infinite-lived agents, but there is land yielding the vector a of goods each period. The infinite-lived agents have sold claims to all of their future endowments to the finite-lived agents for immediate consumption and the finite-lived agents pass these claims down from generation to generation. In the latter case the rate of time discount of the least impatient

infinite-lived agent determines the rate of return for the whole economy.

Either type of steady state, considered alone, might fail to exist. This is easily shown by a simple example that provides an intuitive grasp of the relation between the two types of steady states. Consider the case of a single good and no production activities. A steady state is a β satisfying

$$z(\beta)+y(\beta)+\sum_{h=1}^{H} c^{h}-a=0,$$
$$\beta_1 \leqslant \beta < 1, \quad \text{and}$$
$$(\beta-\beta_h)c_h=0.$$

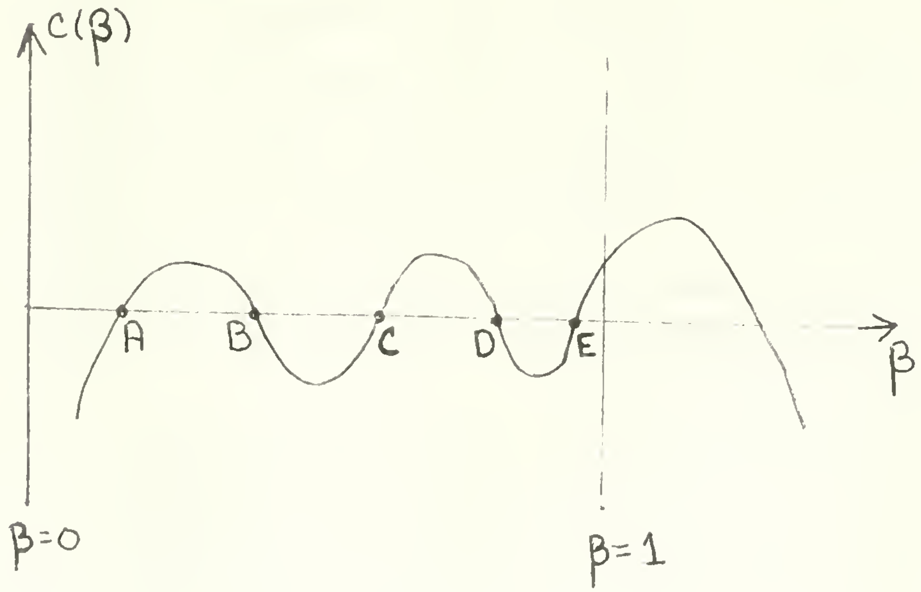

Figure 1.

Define $c(\beta)=a-z(\beta)-y(\beta)$ and plot $c(\beta)$, as shown in Figure 1. For any generation, a steady state β is the price of second period consumption in terms of first period consumption, so the assumptions on excess demand imply that for β near zero or very large, $z(\beta)+y(\beta)$ gets very large. This implies that $c(\beta)$ is negative for β near zero and for β sufficiently large. Walras's law implies that $-\beta z(\beta)=y(\beta)$, so $c(\beta)$ is positive at $\beta=1$. Therefore, there are an odd number of values of β between zero and one for which $c(\beta)=0$. Each such point corresponds to a steady state of the economy consisting of finite-lived agents, no infinite-lived agents, and land yielding the vector a. These values of β are also steady states of the economy with infinite-lived agents whose endowments sum to the vector a, if they satisfy $\beta>\beta_1$. Steady states of the other type are values of β such that $\beta=\beta_1$ and $c(\beta)>0$. Note that in the case of steady states of the second type, the relative amounts of steady state consumption on the part of the agents with $\beta_h=\beta_1$ are completely indeterminate - the c^h's may take any positive values summing to $c(\beta)$.

In the above example neither type of steady state is guaranteed to exist, but one or the other always does. To see this, note that if β_1 is to the right of point E, then a steady state of the first type does not exist, but one of the second type does, and if β_1 is to the left of point E, then a steady state of the first type exists. Muller and Woodford show that

this result is true in general. That is, one of the two types of steady states always exists in the general model. They also show the indeterminacy of allocation of the steady state consumption among infinite-lived consumers with $\beta_h=\beta_1$ is true in general. Let J be the number of infinite-lived consumers with $\beta_h=\beta_1$, and let $\sigma_1,\ldots,\sigma_J$ be an arbitrary set of non-negative weights summing to 1. For any set of weights one chooses, if there exists a steady state with infinite-lived agents consuming, then there exists a steady state in which the relative values of the steady state consumption of these J agents is according to the set of weights $\sigma_1,\ldots,\sigma_J$.

While the development of the general model and the proofs of the existence results are of interest in their own right, they are only the foundation for the main work on determinacy of equilibrium. Following Kehoe and Levine (1982a), Muller and Woodford restrict their attention to equilibrium paths converging asymptotically to the steady state. Kehoe and Levine present a compelling set of arguments justifying such a restriction. They argue that equilibrium paths converging to the steady state are the most plausible perfect foresight equilibria since agents can compute future prices using only local information. Also, note that such a restriction rules out an entire set of equilibria - those not converging to the steady state - so that the resulting concept of determinacy is rather weak. An equilibrium path could be locally unique in

the restricted sense and still belong to a continuum of equilibria. Finally, the equilibrium paths in the neighborhood of the steady state are the easiest to study since familiar mathematical tools can be used.

Muller and Woodford employ the methods of dynamical systems theory to analyze local determinacy. The criteria of local determinacy is familiar from capital theory, where it is often referred to as "saddle-point stability." A steady state x^* of a dynamical system

$$\dot{x} = f(x)$$

has this property if the matrix of derivatives $Df(x^*)$ has exactly the same number of eigenvalues with negative real parts as there are "predetermined" variables in the vector x. That is, if the initial conditions fix initial values for n_1 out of the n elements of x, there must be exactly n_1 eigenvalues of $Df(x^*)$ with negative real parts.

For example, in the familiar case of a two-dimensional system with one predetermined variable (x_1), perfect determinacy requires that $Df(x^*)$ have one eigenvalue with negative real part and one with positive real part, so that x^* is a "saddle point" as shown in Figure 2. In that case, for a given specification $x_1(0)$ of the predetermined variable, there is a single value for x (0), point A in Figure 2, that extends

to a dynamic path converging to the steady state. If both eigenvalues have negative real parts (and are distinct), x^* would be a "stable focus" as shown in Figure 3. In that case, for a given specification $x_1(0)$ of the predetermined variable, $x_2(0)$ can take any of a continuum of values, and still extend to a dynamic path converging to x^*. Thus the equilibrium is indeterminate. And if both eigenvalues have positive real parts (and are distinct), x^* would be an "unstable focus" as shown in Figure 4. In that case no paths converge to x^*, and so most specifications of $x_1(0)$ - any specification other than $x_1(0) = x_1^*$ exactly - are not consistent with any equilibrium path converging to the steady state.

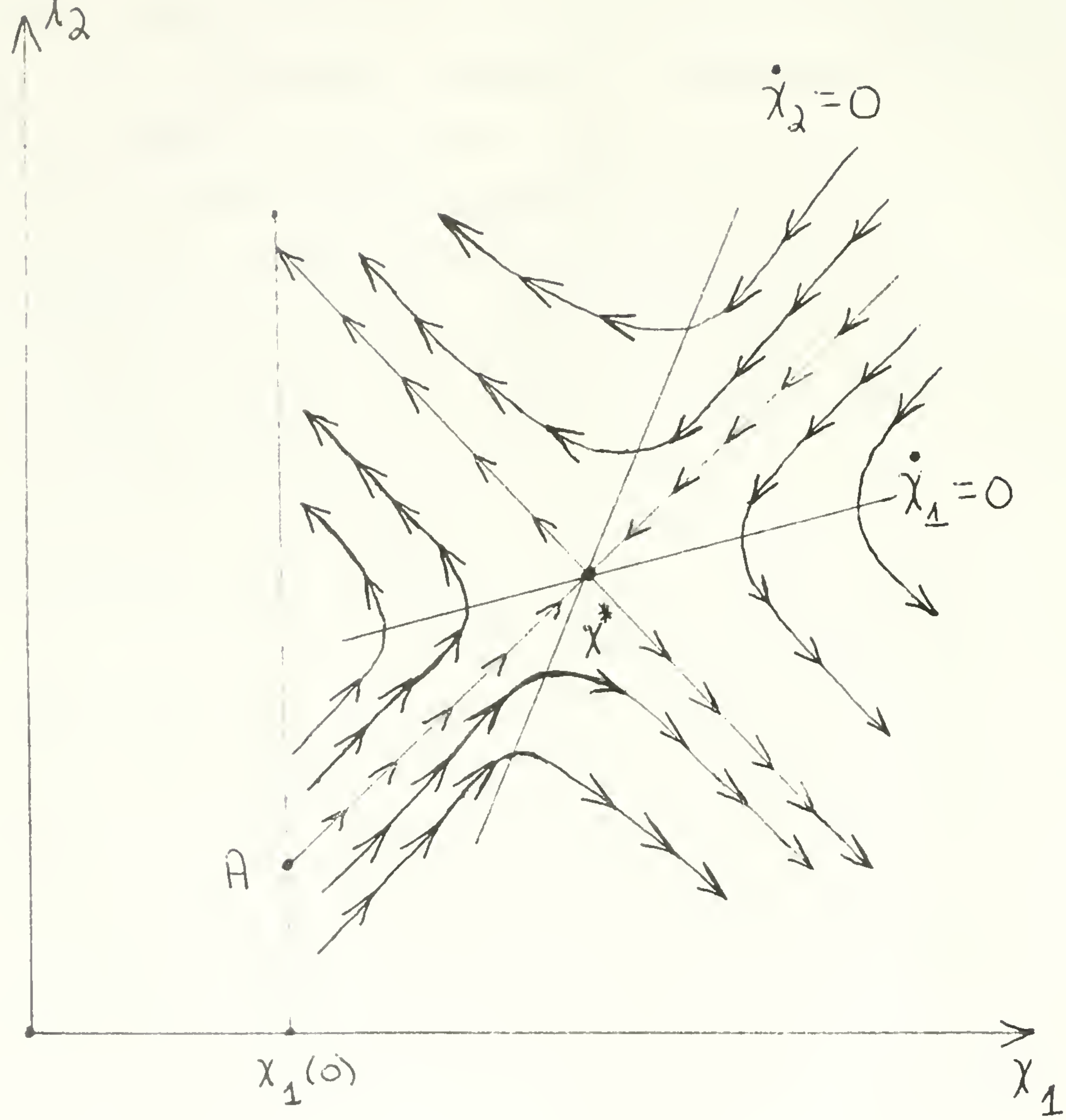

Figure 2. Perfect Determinacy

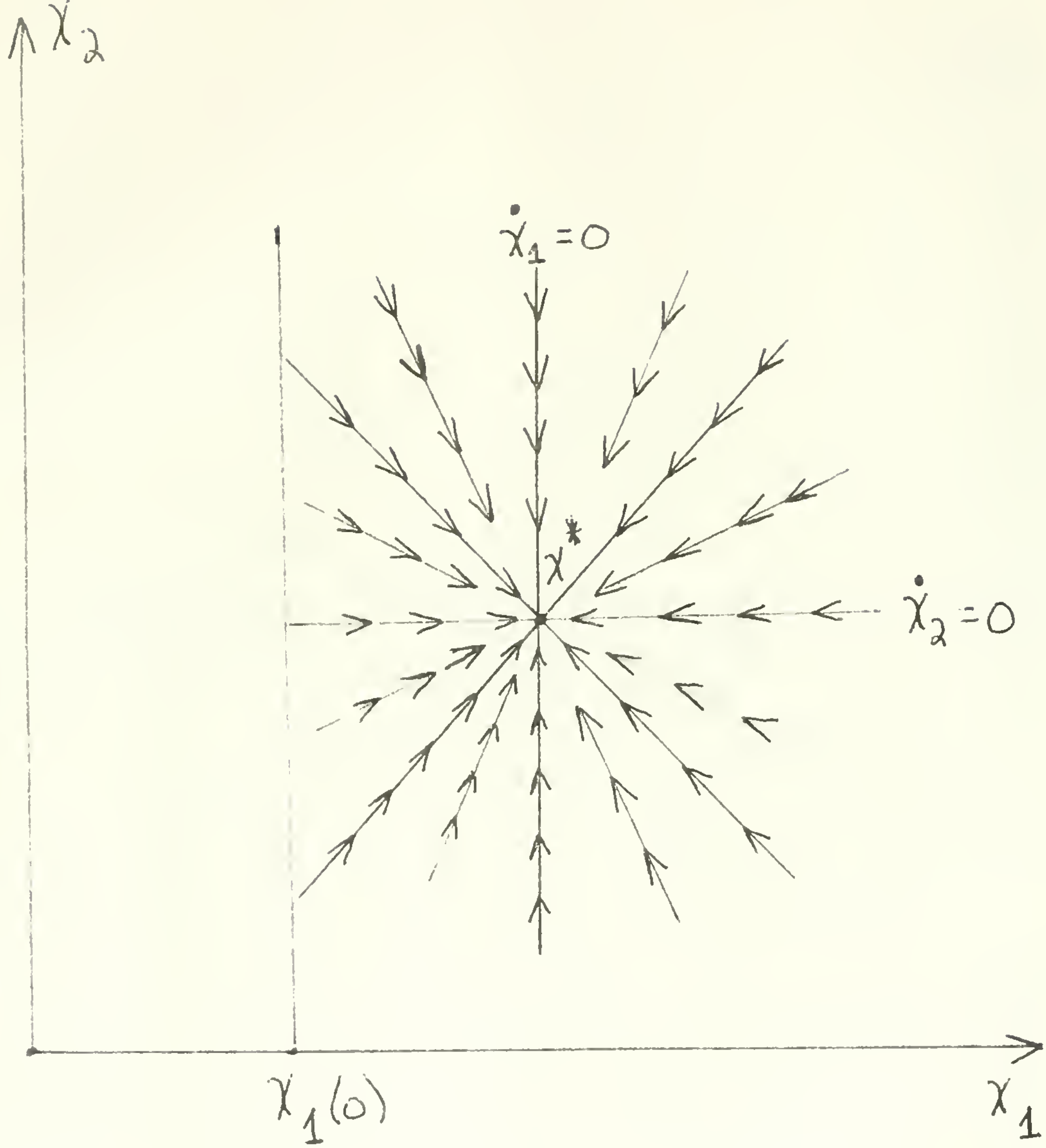

Figure 3. Indeterminacy

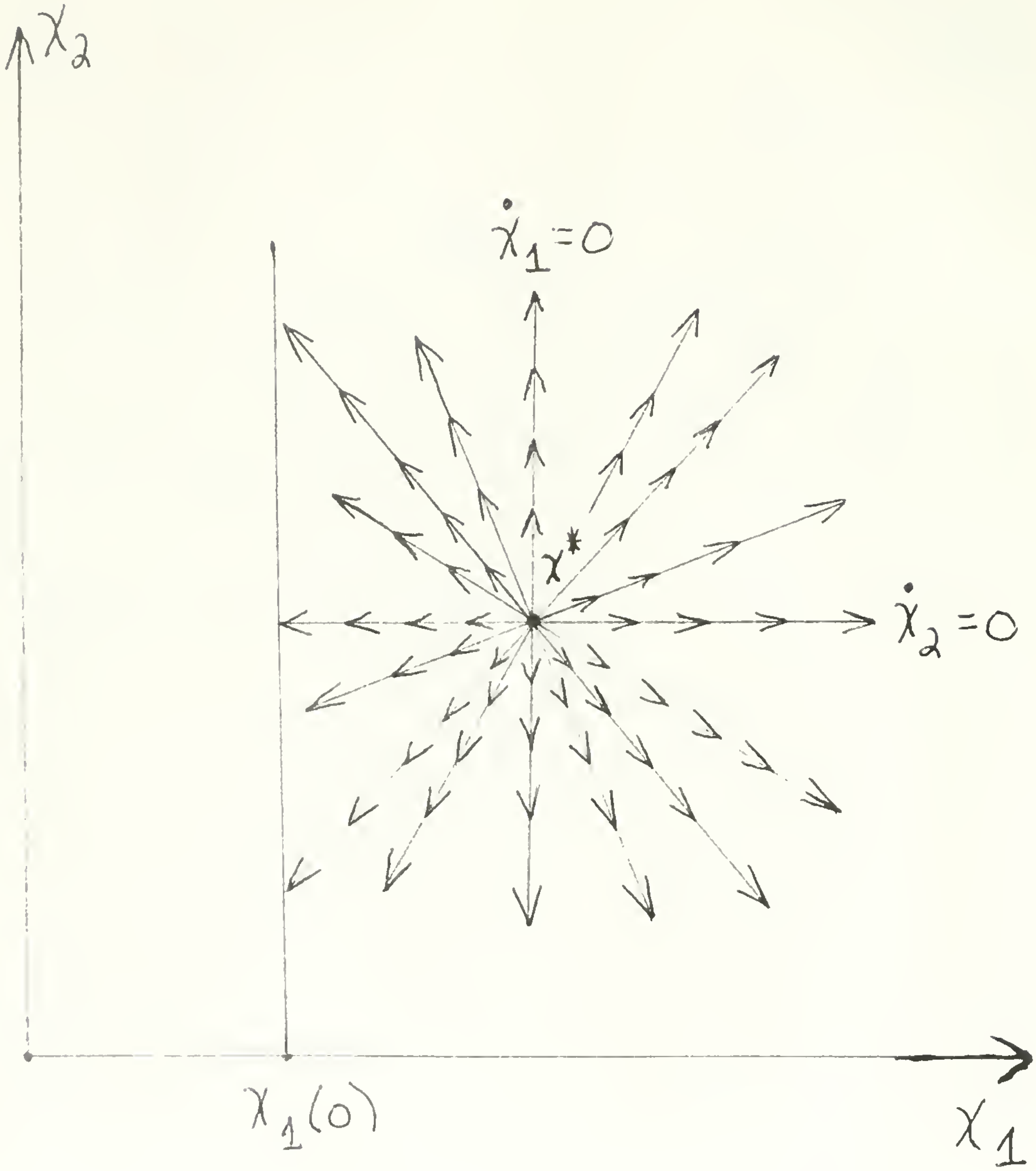

Figure 4. Lack of Structural Stability

The study of discrete time dynamical systems is entirely analogous. Muller and Woodford consider the discrete time dynamical system defined by the equilibrium conditions (2.4) - (2.5). They examine the set of equilibrium paths converging to the steady state and then idenfity the subset of these paths that are consistent with the initial conditions. They show this subset has dimension $\mathrm{Max}(n^s - n, 0)$ where n^s is the number of eigenvalues of the matrix of linearized equilibrium conditions of modulus less than β. Note that in this dynamical system the linearized equilibrium conditions play the same role as $Df(x^*)$ in the capital theory example discussed earlier and the roots of modulus less than β play the same role as the roots with negative real parts. Muller and Woodford show that n^s can range in value from 0 to 2n, so the dimension of the subset of paths converging asymptotically to the steady state can be anywhere from 0 to n. If the dimension is 0, then the equilibrium is either perfectly determinate - corresponding to $n^s = n$ - or not structurally stable - corresponding to $n^s < n$. If the dimension is greater than 0 - $n^s > n$ - then there are a continuum of equilibria converging to the steady state. In this case comparative statics is impossible and perfect foresight seems implausible. Muller and Woodford show that in general, economic theory imposes no restrictions on what the

dimension of the indeterminacy is. That is, for any possible dimension of indeterminacy between 0 and n, there are large sets of economies with that dimension of indeterminacy.

The dimension of indeterminacy depends on n^s, the number of eigenvalues of modulus less than β. Múller and Woodford identify a set of extreme cases in which the relevant eigenvalues split around $\sqrt{\beta}$. That is, there are n eigenvalues of modulus greater than $\sqrt{\beta}$ and n less than $\sqrt{\beta}$. Since $\beta < 1$, this implies that $n^s \leq n$ and that the equilibrium is either perfectly determinate or not structurally stable. These extreme cases are the following:

(i) if the infinite-lived agents are sufficiently more numerous than the finite-lived agents;

(ii) if the sum of the number of independent production activities in use and the number of independent ways in which goods are nearly perfect substitutes for each other is equal to the number of goods in each period; or

(iii) if all finite-lived consumers choose consumption vectors in steady state close to their endowment vectors, or if in the excess demand functions of the finite-lived agents substitution effects are much larger than income effects.

The first extreme case indicates that if infinite-lived agents dominate the economy, then indeterminacy is impossible.

This is a generalization of Kehoe and Levine's (1982a) result that in a pure exchange economy with only infinite-lived agents, equilibria are generically determinate. The second case may seem confusing, but there is a very intuitive justification for it to be true. Indeterminacy occurs when there are too many prices that satisfy the equilibrium conditions. If there are other restrictions on the prices, then these may reduce the possibility of indeterminacy. This is the way to think of condition (ii). If two goods are perfect substitutes, then there has to be a specific relationship between the two good's prices. If a production activity is in use, it must earn zero profit, so each production activity in use puts a restriction on the prices. Condition (ii) says that if there are enough restrictions of this sort, indeterminacy cannot occur. Condition (iii) repeats a familiar theme in economics. That is, that income effects cause problems. Here, if income effects are relatively small, then indeterminacy is not a problem. It follows for indeterminacy to be a problem, the economy must not be close to any of these extremes. This seems to exhaust the conclusions about the conditions for determinacy that may be drawn without restricting one's attention to specific functional forms.

To summarize, Muller and Woodford develop a stationary overlapping generations model with production, with infinite-lived consumers, and with non-depreciating assets. They

establish the existence of perfect foresight equilibria and the existence of steady states. They show that in their model indeterminacy is possible and that the dimension of indeterminacy is limited only by the number of goods in each period. Finally, they show that in certain extreme cases indeterminacy is impossible.

III. The Overlapping Generations Literature

There is a well-developed literature on the overlapping generations model tracing its origins to Samuelson (1958). The work of Muller and Woodford is most closely related to the preliminary treatments of production in the model and the early studies of determinacy of equilibrium. In what follows I review the main papers in the two areas and relate their results to those of Muller and Woodford.

Diamond (1965) has extended the simple overlapping generations model to include a neoclassical production function. He uses this framework to study the effects of national debt on the long run competitive equilibria of the model. While the issues he addresses are not related to those addressed by Muller and Woodford, his model is a special case of their model. This is not obvious at first glance since his population is growing at a constant rate, but Kehoe and Levine (1983a) have shown how to convert such a model into a stationary model of the sort developed by Muller and Woodford.

Gale (1972) also studies an overlapping generations economy with a production sector although his formulation differs from Diamond's in one important respect. He does not require that savings equal real investment in equilibrium. The difference is accounted for by the existence of a constant stock of fiat money in the economy. This allows a wider class of equilibria than those obtained by Diamond. Gale shows that all possible steady states are one of two types. They are either equilibria in which the rate of interest equals the rate of growth - the familiar golden rule monetary steady states - or equilibria in which private wealth equals the value of the stock of capital - the type studied by Diamond. For the special case of a two sector model, Gale examines the stability properties of equilibrium paths.

Gale's work is in the same vein as Muller and Woodford's. Monetary steady states do not exist in their general model because of the presence of infinite-lived agents and land, but Muller and Woodford have shown that the two types of steady states described by Gale do exist in the special case of their model in which there is neither infinite-lived agents nor land. Also his stability analysis coincides with their results in the special case he considers.

In another paper Gale (1973) studies a pure exchange overlapping generations model with a single two period lived consumer in each generation and one good in each period. He

finds that a one dimensional indeterminacy is possible and that this indeterminacy is always associated with either equilibria that are inefficient and have positive amounts of nominal debt or equilibria that are efficient and have negative amounts of nominal debt. In both cases the indeterminate equilibria can be indexed by a single number such as the price of fiat money. Calvo (1978) has also constructed examples in which there is a one dimensional indeterminacy. In his examples the indeterminacy is indexed by the price of the asset existing in the economy.

Balasko, Cass, and Shell (1980) and Balasko and Shell (1980, 1981a, 1981b) have extensively studied pure exchange overlapping generations models. Under reasonably general conditions they have established the existence of perfect foresight Walrasian equilibrium, either with or without fiat money as a store of value. They have also extended the indeterminacy results of Gale (1973) to an economy with many goods in each period in the special case that there is a single agent with a log linear utility function in each generation.

Geanakoplos and Polemarchakis (1983) recognize the possiblity of indeterminacy of perfect foresight equilibria in overlapping generations models. They derive indeterminacy results in simple models with and without production. With these in mind, they develop standard macroeconomic models and analyze the effects of various government policies.

Kehoe and Levine (1982a, 1982b, 1983a, 1983b) have presented the first general treatment of indeterminacy in a pure exchange overlapping generations model. Their work has been the foundation for all of Muller and Woodford's work, so naturally a number of their results are special cases of those of Muller and Woodford. However, they have done a more extensive analysis of the pure exchange case than Muller and Woodford have done with the general model. They have explained the results of Balasko and Shell (1981b) showing that the crucial assumption Balasko and Shell make is that each generation has a two period lived representative consumer with a separable utility function. They also show that the results are not robust to changes in this assumption. In another paper Kehoe and Levine (1983b) develop a regularity theory for their model. They show that almost all economies are regular economies. This is important because the qualitative properties of a regular economy remain the same under small perturbations. Therefore if a regular economy has a given dimension of indeterminacy, then there is an open set of economies close to the given economy that also have that dimension of indeterminacy. Kehoe and Levine prove their results in a model where each generation lives for two periods. However, that agents live for only two periods is not crucial. They show that the same results hold true in models where each generation lives for an arbitrary, finite number of periods.

Finally, Kehoe and Levine (1983a) have also constructed examples - as have Muller and Woodford in the general model - of economies with all possible dimensions of indeterminacy.

Summarizing, there are two branches of the overlapping generations literature that are closely related to the work of Muller and Woodford. Diamond and Gale have presented preliminary treatments of production in an overlapping generations model that are very special cases of Muller and Woodford's model. The early papers on determinacy of equilibrium illustrate that indeterminacy is possible in simple models, but they seem preoccupied with the efficiency properties of the indeterminate equilibria and with the fact that the indeterminate equilibria can be indexed by a single number such as the price of fiat money. Kehoe and Levine show that neither of these things is crucial. They present a general treatment of indeterminacy in the pure exchange case establishing that the dimension of the indeterminacy is limited only by the number of goods in each period. Muller and Woodford's work is a generalization of their results to a model that includes production, infinite-lived agents, and land.

IV. The Optimal Growth Literature

Since the concept of determinacy appears in capital theory, one would suspect that there is a relationship between the results existing in that literature and those of Muller and

Woodford. In what follows, I explain exactly what that relationship is. The approach is to first review the major issues addressed in growth theory, highlighting them by discussing the results of representative papers, and then to interpret Muller and Woodford's results in terms of these issues.

The general formulation of the optimal growth problem is to maximize the present discounted value of some given utility function subject to the constraint that the paths of consumption and the capital goods lie in the feasible set. The typical approach has been to describe the model as a Hamiltonian dynamical system and to use the mathematical theory developed for such systems. Traditional growth theory has been concerned with two major issues. They are the behavior of optimal paths in the neighborhood of a steady state of the system and the global behavior of optimal paths.

Scheinkman (1976) proves results on both local and global stability for the case of a small rate of discount. Drawing on the work of Levhari and Liviatan (1972), he shows that saddle point stability is guaranteed in the neighborhood of an optimal steady state if the rate of discount is close enough to zero. Recall from the discussion of dynamical systems in capital theory that an equilibrium is determinate if the number of stable roots equals the number of predetermined variables. If the number of stable roots is less than that, the equilibrium

is unstable. Scheinkman finds that if the discount rate is close enough to zero, then there is exactly the right number of stable roots for saddle point stability. On the other hand, if the rate of discount is too large, the steady state is unstable and the optimal path does not converge to the steady state even if it is in the neighborhood of the steady state. Scheinkman's global turnpike theorem says that if the discount rate is close enough to zero, then regardless of the initial conditions the optimal path asymptotically approaches the steady state. He establishes his global result by proving that optimal paths must visit the neighborhood of the steady state and then citing the local stability results for such neighborhoods.

A number of authors have sought to generalize the global result of Scheinkman. Cass and Shell (1976) provide an excellent survey of this work. The approach has been to develop conditions on the geometry of the Hamiltonian function that are sufficient to yield results even when the rate of discount is not close to zero.

Muller and Woodford's analysis is local in nature, so one would expect their results to be linked to the local results in this literature, and indeed they are. Consider the special case of their model in which there is one infinite-lived agent and no finite-lived agents. This is their extreme case (i), so the equilibrium is perfectly determinate or not structurally stable depending upon n^s, the number of eigenvalues that are

less than β in modulus. With a slight translation of terms, this is exactly Scheinkman's result. Muller and Woodford's β is equal to $1/(1+\delta)$ where δ is the rate of discount, so if δ is close to zero, then β is close to 1. Muller and Woodford's result is that the roots split around $\sqrt{\beta}$. As β gets close to 1, it also gets close to $\sqrt{\beta}$, so that at some point the roots also split around β. Scheinkman's result can now be interpreted as follows. If the discount rate is close enough to zero, $n^s = n$ and the steady state has the saddle property. If not, $n^s < n$ and the steady state is unstable. Therefore, the local analysis in the growth theory literature is analogous to a special case of Muller and Woodford's determinacy results.

In the discussion above, the demand side of the economy guarantees indeterminacy is impossible. Applying Muller and Woodford's results, this is true of any model of this sort that contains only infinite-lived agents. However, if the demand side of the economy moves away from such a structure, indeterminacy is again possible. This observation yields an interpretation of the paper by Burmeister, Caton, Dobell and Ross (1973). They present an optimal growth model in which there is indeterminacy, but they employ an ad hoc demand specification. In light of Muller and Woodford's results, it must be the case that it would take a population of more than

just infinite-lived agents to generate demand of the type they specify.

A more recent paper that can also be related to this result is Bewley (1982). He develops a model that links general equilibrium theory with capital theory. The distinction between his work and the traditional growth models is that he considers an economy with a finite number of infinite-lived agents instead of a single agent. This generalization complicates the analysis considerably. Bewley proves the existence of equilibrium, the existence of a steady state, and an analogue of Scheinkman's turnpike theorem. However, there can be multiple steady states in Bewley's model, so unlike Scheinkman's turnpike theorem, the initial conditions must be specified so that they are consistent with a given steady state for the optimal path to converge to it asymptotically. While the production sector of Bewley's model is slightly different from that of Muller and Woodford, this should not obscure the obvious relationship between the results. His existence results are analogous to their's. As is the case with Scheinkman's turnpike theorem, Muller and Woodford's extreme case (i) parallels Bewley's turnpike theorem in a neighborhood of the steady state in question. That is, if the discount rate is close enough to zero, saddle point stability is guaranteed and otherwise the steady state is unstable.

V. The Macroeconomics Literature

Indeterminacy of equilibrium paths has also been a topic of research by macroeconomists. They have investigated the problem in both perfect foresight models and rational expectations models. However, most of the models used in macroeconomics are stochastic and therefore cannot be directly linked to the work of Muller and Woodford. The result is that this section is rather short.

In what follows I examine the work done on indeterminacy in perfect foresight models and explain how it relates to the results of Muller and Woodford. Then I discuss the similarities between the problem in perfect foresight models and in rational expectations models. Finally, I mention some of the approaches researchers have taken to deal with indeterminacy of rational expectations paths.

Black (1974), Brock (1974, 1975), and Calvo (1979) have all presented examples of indeterminacy of perfect foresight paths. Here I restrict attention to Calvo's work since it is an extension of Brock's and captures the same ideas as Black's. Calvo considers a simple model with one infinite-lived agent and a production sector. If the production sector were the same as Muller and Woodford's, their results imply that indeterminacy would be impossible. However, in this model real money balances enter the production function, so their results do not apply. Indeed, Calvo shows that indeterminacy

is possible and presents an example in which it occurs. In light of Muller and Woodford's results, it is evident that the indeterminacy that Calvo finds - and that Black and Brock find - is tied to the existence of money in the economy and is not the real relative price indeterminacy discussed by Muller and Woodford. In fact, it is not even necessary to have an intertemporal model to generate this type of indeterminacy. Therefore, techniques that resolve indeterminacy of the sort found by Calvo may or may not have any effect on real relative price indeterminacy.

Burmeister (1980) addresses a number of issues related to the work of Muller and Woodford. He discusses the concept of determinacy in perfect foresight models and rational expectations models where it is assumed that the equilibrium paths converge. He notes that, as in perfect foresight models, the relevant measure of indeterminacy in rational expectations models is the dimension of that subset of the set of all convergent paths, that is consistent with initial conditions. Burmeister also discusses the assumption of convergence of equilibrium paths. He objects to this assumption, citing the possibility of indeterminacy as the main reason. Burmeister's discussion indicates that an analysis of determinacy in rational expectations models may follow an approach similar to that of Muller and Woodford if one is interested in convergent

paths. His discussion also suggests that future researchers should re-examine the assumption of convergence.

The rational expectations literature is full of comments on the multiplicity of equilibrium paths. A number of researchers have discussed criteria for determining a unique solution path. Taylor (1977) reviews some of these and proposes a criterion of his own - that of minimizing the unconditional variance of the price level. Blanchard (1979) also examines the criteria that have been proposed, finding none of them satisfactory. McCallum (1983) develops his own criterion - that of using the minimum set of state variables that suffices to determine the solution path for all admissible parameter values - which he claims determines a unique path for a wide class of models. McCallum (1982) discusses how his criterion singles out a perfect foresight path in the overlapping generations model of Calvo (1978). He shows that it rules out all paths converging asymptotically to the steady state and admits only the steady state path itself.

VI. Summary and Conclusions

Muller and Woodford have developed a very general infinite horizon model of the economy and have shown that indeterminacy of equilibrium is a potential problem in models of the sort they consider. This paper has sought to make their results accessible to those who are interested in the issues addressed

but wish to avoid the technical details of Muller and Woodford's work. The approach has been to first present an intuitive discussion of their work and then to provide a perspective for their results by relating them to the relevant papers in the literature. This survey of the literature has indicated that, prior to the work of Kehoe and Levine, researchers were aware of the possibility of indeterminacy but not the extent to which it could occur.

The discussion in this paper has pointed to a few obvious extensions of the work of Muller and Woodford. The first is to develop a regularity theory for their general model as Kehoe and Levine (1983b) have for the pure exchange case. Another is to consider models in which each generation lives for an arbitrary, finite number of periods and the production technology is generalized accordingly. As Kehoe and Levine have found in their model, this should not change the indeterminacy results, but it would add to the realism of the model. The section of this paper on the Macroeconomics literature points to a sorely needed generalization. That is, is to link Muller and Woodford's results to the rational expectations literature by adding uncertainty to their model.

However, the most important question future researchers face is how to deal with indeterminacy in infinite horizon models. When there is indeterminacy in a model, comparative statics is impossible. Muller and Woodford's results then

indicate that models of the type they develop are not admissible tools for comparative statics analysis unless restricted to the extreme cases listed earlier.

Finally, when indeterminacy is present, perfect foresight is an implausible assumption. It may be the case that the introduction of uncertainty will rectify the problem, but this seems doubtful given the existing rational expectations literature. It is likely that indeterminacy problems will be as prevalent if not more so. This suggests that a direction for future research is toward alternative expectations assumptions. Kehoe and Levine (1982a) have shown that equilibria are determinate in their pure exchange model if agents forecast future prices solely as a function of current prices. It may be the case that expectations do not have to be far from the perfect foresight assumption to get determinacy or that there are theoretically attractive alternatives to the perfect foresight assumption. The results discussed in this paper indicate that these are important topics for future researchers to pursue.

REFERENCES

Y. Balasko, D. Cass, and K. Shell (1980), "Existence of Competitive Equilibrium in a General Overlapping Generations Model," Journal of Economic Theory, 23, 307-322.

Y. Balasko and K. Shell (1980), "The Overlapping Generations Model, I: The Case of Pure Exchange without Money," Journal of Economic Theory, 23, 281-306.

Y. Balasko and K. Shell (1981a), "The Overlapping Generations Model, II: The Case of Pure Exchange with Money," Journal of Economic Theory, 24, 112-142.

Y. Balasko and K. Shell (1981b), "The Overlapping Generations Model, III: The Case of Log-Linear Utility Functions," Journal of Economic Theory, 24, 143-152.

T. Bewley (1982), "An Integration of Equilibrium Theory and Turnpike Theory," Journal of Mathematical Economics, 10, 233-267.

F. Black (1974), "Uniqueness of the Price Level in Growth Models with Rational Expectations," Journal of Economic Theory, 7, 53-65.

O. Blanchard (1979), "Backward and Forward Solutions for Economies with Rational Expectations," American Economic Review, 69, 114-118.

W.A. Brock (1974), "Money and Growth: The Case of Long Run Perfect Foresight," International Economic Review, 15, 750-777.

W.A. Brock (1975), "A Simple Perfect Foresight Monetary Model," Journal of Monetary Economics, 1, 133-150.

E. Burmeister (1980), "On Some Conceptual Issues in Rational Expectations Modelling," Journal of Money, Credit, and Banking, 12, 800-816.

E. Burmeister, C. Caton, R. Dobell, and S. Ross (1973), "The Saddlepoint Property and the Structure of Dynamic Heterogeneous Capital Good Models," Econometrica, 41, 79-95.

G. Calvo (1978), "On the Indeterminacy of Interest Rates and Wages with Perfect Foresight," Journal of Economic Theory, 19, 321-337.

G. Calvo (1979), "On Models of Money and Perfect Foresight," International Economic Review, 20, 83-103.

D. Cass and K. Shell (1976), "Introduction to Hamiltonian Dynamics in Economics, Journal of Economic Theory, 12, 1-10.

P.A. Diamond (1965), "National Debt in a Neoclassical Growth Model," American Economic Review, 55, 1126-1150.

D. Gale (1972), "On Equilibrium Growth of Dynamic Models," in: Mathematical Topics in Economic Theory and Computations, Society for Industrial and Applied Mathematics, Philadelphia, Pennsylvania, 89-98.

D. Gale (1973), "Pure Exchange Equilibrium of Dynamic Economic Models," Journal of Economic Theory, 6, 12-36.

J.D. Geanakoplos and H.M. Polemarchakis (1983), "Walrasian Indeterminacy and Dynamic Macroeconomic Equilibrium: The Case of Certainty," unpublished manuscript.

T.J. Kehoe and D.K. Levine (1982a), "Comparative Statics and Perfect Foresight in Infinite Horizon Economies," M.I.T. Working Paper #312.

T.J. Kehoe and D.K. Levine (1982b), "Intertemporal Separability in Overlapping Generations Models," M.I.T. Working Paper #315.

T.J. Kehoe and D.K. Levine (1983a), "Indeterminacy of Relative Prices in Overlapping Generations Models," M.I.T. Working Paper #313.

T.J. Kehoe and D.K. Levine (1983b), "Regularity in Overlapping Generations Exchange Economies," M.I.T. Working Paper #314.

D. Levhari and N. Liviatan (1972), "On Stability in the Saddle-point Sense," Journal of Economic Theory, 4, 88-93.

B.T. McCallum (1982), "The Role of Overlapping-Generations Models in Monetary Economics," N.B.E.R. Working Paper #989.

B.T. McCallum (1983), "On Non-Uniqueness in Rational Expectations Models," Journal of Monetary Economics, 11, 139-168.

W.J. Muller, III and M. Woodford (1983), "Stationary Overlapping Generations Economies with Production and Infinite-Lived Consumers: I. Existence of Equilibrium," M.I.T. Working Paper #325.

W.J. Muller, III and M. Woodford (1983), "Stationary Overlapping Generations Economies with Production and Infinite-Lived Consumers: II. Determinacy of Equilibrium," M.I.T. Working Paper #326.

P.A. Samuelson (1958), "An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money," Journal of Political Economy, 6, 467-482.

J. Scheinkman (1976), "On Optimal Steady States of n-Sector Growth Models when Utility is Discounted," Journal of Economic Theory, 12, 11-30.

J.B. Taylor (1977), "Conditions for Unique Solutions in Stochastic Macroeconomic Models with Rational Expectations," Econometrica, 45, 1377-1386.

J. Tirole (1983), "Asset Bubbles and Overlapping Generations: A Synthesis," CERAS Working Paper.

C.A. Wilson (1981), "Equilibrium in Dynamic Models with an Infinity of Agents," Journal of Economic Theory, 24, 95-111.

Bar Code is on
Last Page